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LETTER TO THE EDITOR

A particular class of Einstein spaces

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Abstract. A hierarchy of special solutions of the vacuum field equations of general relativity in dimensions $n \geqslant 5$ is explicitly constructed from the solutions for $n=4$. The basic feature of these solutions is that they admit a non-trivial 0-analysis. The meaning of this particular constraint and its relation to plane gravitational waves is also discussed.

An interesting class of pseudo-Riemannian spaces, the projective properties of which are very close to those of the spaces of constant curvature, were extensively studied in the early 1960s by a school of Russian Mathematicians [1–3]. These spaces, are the $V(K)$-spaces which are *semireducible* Riemannian or pseudo-Riemannian spaces, the fundamental form of which can be written, in a suitably chosen coordinate system, as

$$ds^2 = ds_0^2 + \sum_{\alpha=1}^{p} \sigma_{(\alpha)}^2 \, ds_\alpha^2 ; \tag{1}$$

where

$$ds_0^2 = g_{i_0 j_0} \, dx^{i_0} \, dx^{j_0} \qquad (i_0, j_0 = 1 \ldots l) \tag{1a}$$

$$ds_\alpha^2 = g_{i_\alpha j_\alpha} \, dx^{i_\alpha} \, dx^{j_\alpha} \qquad (i_\alpha, j_\alpha = 1 \ldots n_\alpha, \alpha = 1, \ldots p) \tag{1b}$$

and

$$\sigma_{(\alpha)} = \sigma_{(\alpha)}(x^{i_0}) \qquad (\sigma_{(\alpha)}/\sigma_{(\beta)} \neq \text{const.} (\alpha \neq \beta)). \tag{1c}$$

Here, the principal part of the metric ds_0^2 and the functions $\sigma_{(\alpha)}$ are such that the *adjoint* metric [4]

$$ds_*^2 = ds_0^2 + \sum_{\alpha=1}^{p} \sigma_{(\alpha)}^2 (dy^\alpha)^2 \tag{2}$$

is that of a space of constant curvature K. It is worth noticing that the additional metrics ds_α^2 in (1) are quite arbitrary, the definition of a $V(K)$-space depending solely on the properties of the analysis (1) as well as on the value of the constant K. We further assume that $l \geqslant 1$ and $n_\alpha > 1$ and we notice that the Levi–Civita spaces for which *not* all the roots are multiple can be written in the form (1). However, a Levi–Civita space is *not* in general a $V(K)$-space. The characteristic geometric property of a $V(K)$-space is that through each geodesic of the space there is a totally geodesic hypersurface of constant curvature K, such that the induced metric on it, in the coordinate system where the K-analysis (1) holds, is given by (2) [2(1967)].

The analysis (1) of a $V(K)$-space is characteristic to the space under consideration and it is called the *K-analysis* of the space. Thus, any other analysis, say K_*, in a different coordinate system implies $K_* = K$ [1]. The K-analysis is said to be *maximal*, if none of the metrics ds_α^2 is that of a $V(K_\alpha)$-space. Since the maximal K-analysis is unique up to trivial transformations [1(1958)], the representation (1) is also unique up to trivial transformations. Each K-analysis is also characterized by the *conjugate curvatures* of the ds_α^2, which are the quantities

$$K_\alpha = \Delta_1 \sigma_{(\alpha)} + K\sigma_{(\alpha)}^2 \qquad (\alpha = 1, \dots, p) \tag{3}$$

where Δ_1 is the differential parameter of the first order. It can be easily proved that the conjugate curvatures of a $V(K)$-space are constants [3]. Besides, if the metric ds_α^2 (for some fixed value of α) admits a K_α-analysis and ds_β^2 is an additional metric in the aforementioned K_α-analysis then the conjugate curvature of the metric ds_β^2 is the same in both the K_α-analysis of the metric ds_α^2 and in the resulting K-analysis of the original $V(K)$-space. In particular, the constants K_α give a rough measure of how much a $V(K)$-space differs (locally) from a space of constant curvature K of the same dimensionality. In fact, it can be proved that the space (1) is of constant curvature K, iff each metric ds_α^2 is of constant curvature K_α [1]. This particular property of the K-analysis is essential for our subsequent analysis.

Although the spaces $V(K)$ have been exhaustively studied, at least as far as their local properties are concerned, to the best of our knowledge, the *Einstein spaces* which are also $V(K)$-spaces have not been considered. Now, using the representation (1) and the condition that the adjoint metric (2) is of constant curvature K, we can easily calculate the Ricci tensor of the arbitrary space $V(K)$, namely

$$R_{i_0 j_0} = K(n-1) g_{i_0 j_0} \tag{4a}$$

$$R_{i_\alpha j_\alpha} = \underset{(\alpha)}{R}_{i_\alpha j_\alpha} + K(n-1) \underset{(\alpha)}{\sigma}{}^2 g_{i_\alpha j_\alpha} - K_\alpha (n_\alpha - 1) g_{i_\alpha j_\alpha} \tag{4b}$$

$$R_{i_\alpha j_0} = R_{j_\beta k_\alpha} = 0 \qquad (\alpha \neq \beta). \tag{4c}$$

Assuming that the metric ds_α^2 is that of an Einstein space which is *not* a space of constant curvature, i.e.

$$\underset{(\alpha)}{R}_{i_\alpha j_\alpha} = K_\alpha (n_\alpha - 1) g_{i_\alpha j_\alpha} \tag{5}$$

we get from (4*a*, *b*, *c*) that (1) is the metric of an Einstein space, which is not a space of constant curvature because of our previous remark. Conversely, if (1) corresponds to a non-trivial Einstein space then from (4*b*) we obtain condition (5), which implies that the metric ds_α^2 is that of an Einstein space, for each $\alpha = 1, \dots, p$. Besides, since the resulting Einstein space is non-trivial and K_α is the conjugate curvature of ds_α^2 in the analysis (1) one, at least, of the Einstein spaces ds_α^2 is non-trivial and consequently $n_\alpha \geqslant 4$ for at least one value of the index α. Thus, we have proved the following.

Theorem. The necessary and sufficient conditions that a $V(K)$-space is a non-trivial Einstein space are

(i) Each metric ds_α^2 in the K-analysis (1) is that of an Einstein space of scalar curvature $n_\alpha(n_\alpha - 1)K_\alpha$, where K_α is the conjugate curvature of ds_α^2.

(ii) One at least of the above Einstein spaces is non-trivial (i.e. not a space of constant curvature).

Since the dimension of a $V(K)$-space is

$$n=1+\sum_{\alpha=1}^{p} n_\alpha \tag{6}$$

we have, as an immediate consequence of the above theorem, that the dimension n, of a $V(K)$-space which is also an Einstein space is necessarily $n \geqslant 5$. This means that we need at least five dimensions in order to be able to distinguish between a space of constant curvature and an Einstein space which is also a $V(K)$-space.

We specialize now to the case $K=0$, where the corresponding spaces $V(0)$ can be regarded as solutions of the vacuum field equations of general relativity in a spacetime of $n \geqslant 5$ dimensions. In this case the *principal part* of the metric (1), ds_0^2, is necessarily flat and the $\sigma_{(\alpha)}$ are linear combinations of the x^{i_0}[1, 3]. Besides, a complete classification of the $V(0)$-spaces has been given by Kruckovic for any possible signature of the metrics involved [3]. However, this classification scheme is not particularly convenient, since it treats (locally) decomposable spaces $V(0)$ and the irreducible (non-decomposable) ones in the same way. Thus, removing the redundancies of the aforementioned classification scheme and concentrating only on *non-decomposable metrics* we conclude with two essentially distinct classes of $V(0)$-spaces the members of which are completely specified (up to a choice of the constants involved in the expressions of the $\sigma_{(\alpha)}$) by integers. In particular we have

$$\textit{type I}: l=p=1 \qquad K_1=1 \qquad (n_1 \geqslant 4) \tag{7a}$$

$$\textit{type II}: l=2\rho \qquad p \geqslant 1 \qquad K_\alpha=0 \qquad (\alpha=1,\ldots,p \qquad \rho \geqslant 1) \tag{7b}$$

where $n_\alpha \geqslant 4$ for at least one value of α $(1 \leqslant a \leqslant p)$. Now the signature of the flat metric ds_0^2 is uniquely prescribed, being ρ, for any $\rho \geqslant 1$. Thus, the explicit form of the metrics of type II depends only on the structure of the $\sigma_{(\alpha)}$ which is not essential for our discussion. Furthermore, the requirement of a Lorentz signature for the spacetime under consideration fixes the values of ρ to $\rho=1$, the additional metrics now being necessarily positive definite $(n \geqslant 6)$.

An immediate result of our analysis is that the solutions of the vacuum field equations which admit a 0-analysis in dimension $n=5$, can be uniquely prescribed from the solutions in dimension $n=4$, which have a cosmological constant $\Lambda=K_1=1$ and vice-versa. In exactly the same way the solutions which admit a 0-analysis in dimension $n=6$ are uniquely prescribed by the 4-dimensional Einstein spaces for which $R=K_1=0$ and vice versa. Clearly, the discriminating factor between the above cases is the conjugate curvature K_1 which is different for each type of metric. In this sense it would be interesting to examine cases with $K \neq 0$. However, since there is no apparent physical meaning to the requirement of a K-analysis on the solutions of the Einstein field equations the understanding of this particular constraint seems more adequate. In a certain sense, requiring that an Einstein space is a $V(K)$-space means that we endow our spacetime with additional symmetry and in particular with *projective symmetry* the maximal content of which is encountered in spaces of constant curvature. However, to accommodate this extra symmetry in four dimensions seems to be very stringent and an increase of the dimensionality of the background manifold is unavoidable.

The spaces $V(0)$ of the type II_p $(p \geqslant 1)$ have another interesting property, namely they admit *absolutely parallel* vector fields. Hence the solutions of the vacuum field equations of this particular type satisfy the criterion of Kundt for plane gravitational waves [6]. Clearly, there is a *hierarchy* of such solutions which can be constructed from the solutions in dimension $n=4$. In particular, for a solution of the type II_p this construction is uniquely prescribed in dimension $n=2+4p$. In fact, there are *no* solutions of type II_p in dimensions $n<2+4p$, since if such a solution exists, one of the additional metrics should have a dimension $n_\alpha<4$, being necessarily flat. In this case this particular term of the K-analysis can be absorbed in the principal part of the metric leading into a (locally) decomposable space which contradicts the definition of a space of type II_p.

Last, but not least, the process which generates the aforementioned hierarchy of solutions for any $p\geqslant 1$ can be regarded as a concrete realization of a relativistic extension scheme suggested by the author in a different context [5]. In fact spaces of type I are extensions of $\Gamma_\pm$-type while spaces of type II_p can be regarded as singular extensions of the $B_\pm$-type of the aforementioned reference.

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